The first integral method for solving some nonlinear equations

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Abstract

In this paper, the first integral method is used to solve the combined Kortewge-de Vries-modified Kortewege-de Vries(Kdv-mKdv), the (2 + 1)-dimensional Broer-Kaup(BK), the Burgers-Huxley and the Newwell-Whitehead equations. New multiple traveling wave solutions are obtained for these equations. It is shown that the proposed method is direct and effective.

Keywords: First integral method; combined Kdv-mKdv equation; BK equation; Burgers-Huxley equation; Newwell-Whitehead equation; Solitary wave solution.

Mathematics Subject Classification : 53A15

1 Introduction

Most phenomena in real world are described through nonlinear equations. In the recent decades, many effective methods for obtaining exact solutions of nonlinear evolution equations (NLEEs) have been presented. These methods include Painleve method [11], Jacobi elliptic function method [7], Hirota’s bilinear method [6], the sine-cosine function method [10], the tanh-coth function method [3], the exp-function method [5], the \( \frac{G'}{G} \)-expansion method [8], the homogeneous balance method [9] and so on.

Recently, Zayed and Gepreel [12] used the \( \frac{G'}{G} \)-expansion method to obtain exact solutions of the combined Kortewgec-de Vries-modified Kortewgec-de Vries equation. Also, by using this method, Zhang, Wei and Lu [13] reported new exact solutions of the (2 + 1)-dimensional Broer-Kaup equation.

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The first integral method, which is based on the ring theory of commutative algebra, was first proposed by Z. Feng [4]. This method was further developed by the same author and some other mathematicians.

In this work, we apply the first integral method to solve the combined Korteweg-de Vries-modified Korteweg-de Vries, the $(2 + 1)$-dimensional Broer-Kaup, the Burgers-Huxley and the Newwell-Whitehead equations.

The remaining structure of this article is organized as follows: Section 2 is a brief introduction to the first integral method. In section 3, by implementing the first integral method, some new exact solutions for the combined Korteweg-de Vries-modified Korteweg-de Vries equation are reported. The first integral method is used to solve the $(2 + 1)$-dimensional Broer-Kaup equation in section 4. In sections 5 and 6 by using this method, some exact solutions for the Burgers-Huxley and the Newwell-Whitehead equations are reported, respectively. A conclusion is summarized in the last section.

2 The first integral method

Consider a general nonlinear partial differential equation (PDE) in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0.$$  \hfill (1)

Using the wave variable $\xi = x - ct$ carries (1) into the following ordinary differential equation (ODE)

$$Q(U, U', U'', \ldots) = 0,$$  \hfill (2)

where prime denotes the derivative with respect to the variable $\xi$.

Next, we introduce new independent variables $x = u$, $y = u_\xi$ which change (2) to a system of ODEs

$$\begin{cases}
x' = y, \\
y' = f(x, y).
\end{cases}$$  \hfill (3)

According to the qualitative theory of ordinary differential equation [2], if one can find two first integrals to system (3) under the same conditions, then analytic solutions to (3) can be solved directly. However, in general, it is really difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how to find its first integrals in a systematic way, nor is there a logical way for telling us what these first integrals are. A key idea of the paper to find the first integral is to utilize the Division Theorem. For convenience, first let us recall the Division Theorem for two variables in the complex domain $\mathbb{C}$ [1].

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Theorem 2.1 (Division Theorem.) Suppose that $P(x,y)$ and $Q(x,y)$ are polynomials of two variables $x$ and $y$ in $\mathbb{C}[x,y]$ and $P(x,y)$ is irreducible in $\mathbb{C}[x,y]$. If $Q(x,y)$ vanishes at all zero points of $P(x,y)$, then there exists a polynomial $G(x,y)$ in $\mathbb{C}[x,y]$ such that

$$Q(x,y) = P(x,y)G(x,y).$$

3 Exact solution for combined Kdv-mKdv equation

Consider the combined Kdv-mKdv equation

$$u_t + \alpha uu_x + \beta u^2 u_x + u_{xxx} = 0, \quad (4)$$

where $\alpha$ and $\beta$ are nonzero real constants. Assume that Eq. (4) has an exact solution in the form

$$u(x, t) = u(\xi), \quad \xi = x - ct, \quad (5)$$

where $c$ is real constant to be determined. By substituting (5) into Eq. (4), we obtain

$$-cu' + \alpha uu' + \beta u^2 u' + u''' = 0, \quad (6)$$

where prime denotes the derivative with respect to the variable $\xi$. Integrating Eq. (6) with respect to $\xi$ and considering the constant of integration to be zero, we have

$$-cu + \frac{\alpha}{2} u^2 + \frac{\beta}{3} u^3 + u'' = 0. \quad (7)$$

Next, we introduce new independent variables $x = u$, $y = u_{\xi}$ which changes (7) to a system of ODEs

$$\begin{cases}
  x' = y, \\
  y' = cx - \frac{\alpha}{2} x^2 - \frac{\beta}{3} x^3.
\end{cases} \quad (8)$$

Now, we apply the Division theorem to seek the first integral to (8). Suppose that $x - x(\xi)$ and $y - y(\xi)$ are the nontrivial solutions to (8), and $p(x, y) = \sum_{i=0}^{m} a_i(x) y^i$, is an irreducible polynomial in $\mathbb{C}[x, y]$ such that

$$p(x(\xi), y(\xi)) = \sum_{i=0}^{m} a_i(x(\xi)) y(\xi)^i = 0, \quad (9)$$
where \(a_i(x)\) \((i = 0, \ldots, m)\) are polynomials of \(x\) and \(a_m(x) \neq 0\). Eq. (9) would also be the first integral to (8). We start our study by assuming \(m = 1\) in (9). Note that \(\frac{dp}{d\xi}\) is a polynomial in \(x\) and \(y\), and \(p[x(\xi), y(\xi)] = 0\) implies \(\frac{dp}{d\xi}(8) = 0\). By the Division Theorem, there exists a polynomial \(H(x, y) = h(x) + g(x)y\) in \(\mathbb{C}[x, y]\) such that

\[
\frac{dp}{d\xi}(8) = \left(\frac{\partial p}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y} \frac{\partial y}{\partial \xi}\right)(8)
\]

\[
= \sum_{i=0}^{1} a_i'(x)y^{i+1} + \sum_{i=0}^{1} ia_i(x)y^{i-1}(cx - \frac{\alpha}{2}x^2 - \frac{\beta}{3}x^3)
\]

\[
= (h(x) + g(x)y) \left(\sum_{i=0}^{1} a_i(x)y^i\right).
\]

where prime denote differentiating with respect to the variable \(x\). On equating the coefficients of \(y^i\) \((i = 2, 1, 0)\) on both sides of (10), we have

\[
a'_1(x) = g(x)a_1(x),
\]

\[
a'_0(x) = h(x)a_1(x) + g(x)a_0(x),
\]

\[
a_1(cx - \frac{\alpha}{2}x^2 - \frac{\beta}{3}x^3) - h(x)a_0(x).
\]

Since, \(a_1(x)\) is a polynomial of \(x\), from (11) we conclude that \(a_1(x)\) is a constant and \(g(x) = 0\). For simplicity, we take \(a_1(x) = 1\), and balancing the degrees of \(h(x)\) and \(a_0(x)\) we conclude that \(\deg h(x) = 1\), only. Now, suppose that \(h(x) = Ax + B\), then from (12) we find

\[
a_0(x) = \frac{A}{2}x^2 + Bx + D,
\]

where \(D\) is an arbitrary integration constant. Substituting \(a_0(x), a_1(x)\) and \(h(x)\) in (13) and setting the coefficient of powers \(x\) to be zero, then yields

\[
\begin{aligned}
\frac{A^2}{2} &= -\frac{\beta}{3}, \\
\frac{3AB}{2} &= -\frac{\alpha}{2}, \\
AD + B^2 &= c,
\end{aligned}
\]

\[
BD = 0.
\]

(14)
By assuming $\beta < 0$ and solving (14), we obtain

$$A = \sqrt{-\frac{2\beta}{3}}, \quad B = -\alpha \sqrt{-\frac{1}{6\beta}}, \quad D = 0, \quad c = -\frac{\alpha^2}{6\beta}, \quad (15)$$

$$A = -\sqrt{-\frac{2\beta}{3}}, \quad B = \alpha \sqrt{-\frac{1}{6\beta}}, \quad D = 0, \quad c = -\frac{\alpha^2}{6\beta}. \quad (16)$$

From (16), it is concluded $c > 0$. Setting (15) and (16) in (9), we obtain

$$y + \sqrt{-\frac{\beta}{6}} x^2 - \alpha \sqrt{-\frac{1}{6\beta}} x = 0,$$

$$y - \sqrt{-\frac{\beta}{6}} x^2 + \alpha \sqrt{-\frac{1}{6\beta}} x = 0, \quad (17)$$

respectively. Combining equations (17) with (8), we find the exact solutions to equation (7) as follows

$$u_1(\xi) = \frac{\alpha}{-\beta + c_1 \alpha e^{\frac{\alpha}{\sqrt{6\beta}} \xi}},$$

$$u_2(\xi) = \frac{\alpha}{-\beta + c_1 \alpha e^{\frac{-\alpha}{\sqrt{6\beta}} \xi}},$$

where $c_1$ is an arbitrary constant. Then the exact solutions to (4) can be written as

$$u_1(x, t) = \frac{\alpha}{-\beta + c_1 \alpha e^{\frac{-\alpha}{\sqrt{6\beta}} (x - \frac{\alpha^2}{6\beta} t)}},$$

$$u_2(x, t) = \frac{\alpha}{-\beta + c_1 \alpha e^{\frac{\alpha}{\sqrt{6\beta}} (x - \frac{\alpha^2}{6\beta} t)}}.$$

We get distinctive solutions by giving different values to $c_1$.

4 Exact solution for (2+1) dimensional BK system

We consider the (2+1) dimensional BK system

$$\begin{cases} u_{yt} + 2v_{xx} + 2(uu_x)_y - u_{xyy} = 0, \\
v_t + 2(uv)_x + v_{xx} = 0. \end{cases} \quad (18)$$

Assume that system (18) has exact solutions in the form

$$u(x, y, t) = u(\xi), \quad v(x, y, t) = v(\xi), \quad \xi = x + y - ct, \quad (19)$$
where \( c \) is real constant to be determined. By substituting (19) into Eq. (18), we obtain

\[
\begin{align*}
-cu'' + 2v'' + 2(uu')' - u''' &= 0, \\
-cv' + 2(uv)' + v'' &= 0.
\end{align*}
\] (20)

where prime denotes the derivative with respect to the variable \( \xi \). Integrating equations of (20) with respect to \( \xi \) and considering the constant of integration to be zero, we have

\[
\begin{align*}
-cu' + 2v' + 2(uu') - u'' &= 0, \\
-cv + 2(uv) + v' &= 0.
\end{align*}
\] (21)

By the first equation of (21), we have

\[
v' = \frac{u''}{2} - uu' + \frac{cu'}{2}.
\] (22)

Integrating (22) and considering the constant of integration to be zero, yields

\[
v = \frac{u'}{2} - \frac{u^2}{2} + \frac{cu}{2}.
\] (23)

Substitution of (22) and (23) into the second equation of (21) yields

\[
u'' + 3cu^2 - c^2 u - 2u^3 = 0.
\] (24)

Next, we introduce new independent variables \( x = u, y = u_\xi \) which changes (24) to a system of ODEs

\[
\begin{align*}
x' &= y, \\
y' &= 2x^3 - 3cx^2 + c^2 x.
\end{align*}
\] (25)

Now, we apply the Division theorem to seek the first integral to (25). Suppose that \( x = x(\xi) \) and \( y = y(\xi) \) are the nontrivial solutions to (25), and \( p(x, y) = \sum_{i=0}^{m} a_i(x) y^i \), is an irreducible polynomial in \( \mathbb{C}[x, y] \) such that

\[
p(x(\xi), y(\xi)) = \sum_{i=0}^{m} a_i(x(\xi)) y(\xi)^i = 0,
\] (26)
where $a_i(x)$ ($i = 0, \cdots, m$) are polynomials of $x$ and $a_m(x) \neq 0$. We start our study by assuming $m = 1$ in (26). By the Division Theorem, there exists a polynomial $H(x, y) = h(x) + g(x)y$ in $\mathbb{C}[x, y]$ such that

$$
\frac{dp}{d\xi}|_{(25)} = \left( \frac{\partial p}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y} \frac{\partial y}{\partial \xi} \right)|_{(25)}
$$
$$
= \sum_{i=0}^{1} a_i'(x)y^{i+1} + \sum_{i=0}^{1} ia_i(x)y^{i-1}(2x^3 - 3cx^2 + c^2x) \tag{27}
$$
$$
= (h(x) + g(x)y) \left( \sum_{i=0}^{1} a_i(x)y^i \right),
$$

where prime denote differentiating with respect to the variable $x$. On equating the coefficients of $y^i$ ($i = 2, 1, 0$) on both sides of (27), we have

$$
a_1'(x) = g(x)a_1(x), \tag{28}
$$
$$
a_0'(x) = h(x)a_1(x) + g(x)a_0(x), \tag{29}
$$
$$
a_1(2x^3 - 3cx^2 + c^2x) = h(x)a_0(x). \tag{30}
$$

Since, $a_1(x)$ is a polynomial of $x$, from (28) we conclude that $a_1(x)$ is a constant and $g(x) = 0$. For simplicity, we take $a_1(x) = 1$, and balancing the degrees of $h(x)$ and $a_0(x)$ we conclude that $\deg h(x) = 1$, only. Now, suppose that $h(x) = Ax + B$, then from (29) we find

$$
a_0(x) = \frac{A}{2} x^2 + Bx + D,
$$

where $D$ is an arbitrary integration constant. Substituting $a_0(x), a_1(x)$ and $h(x)$ in (30) and setting the coefficient of powers $x$ to be zero, then yields

$$
\begin{align*}
\frac{A^2}{2} &= 2, \\
\frac{3AB}{2} &= -3c, \\
c^2 &= AD + B^2, \\
BD &= 0.
\end{align*} \tag{31}
$$

Solving (31), we obtain

$$
A = 2, \quad B = -c, \quad D = 0, \tag{32}
$$
$$
A = -2, \quad B = c, \quad D = 0, \tag{33}
$$

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where \( c \) is an arbitrary number. Setting (32) and (33) in (26), we obtain

\[
\begin{align*}
y + x^2 - cx &= 0, \\
y - x^2 + cx &= 0,
\end{align*}
\]

respectively. Combining equations (34) with (25), we find the exact solutions to equation (24) as follows

\[
\begin{align*}
u_1(\xi) &= \frac{c}{1 + cc_1e^{-c\xi}}, \\
u_2(\xi) &= \frac{c}{1 + cc_1e^{c\xi}},
\end{align*}
\]

where \( c_1 \) is an arbitrary constant. Combining (35) with (23), we obtain the exact solutions to \( v(\xi) \) as

\[
\begin{align*}
v_1(\xi) &= \frac{c}{(1 + cc_1e^{-c\xi})^2}, \\
v_2(\xi) &= 0,
\end{align*}
\]

respectively. Then the exact solutions to (18) can be written as

\[
\begin{align*}
u_1(x, y, t) &= \frac{c}{1 + cc_1e^{-c(x+y-ct)}}, \\
v_1(x, y, t) &= \frac{c}{(1 + cc_1e^{-c(x+y-ct)})^2},
\end{align*}
\]

and

\[
\begin{align*}
u_2(x, y, t) &= \frac{c}{1 + cc_1e^{c(x+y-ct)}}, \\
v_2(x, y, t) &= 0.
\end{align*}
\]

In (37) and (38), \( c \) and \( c_1 \) are arbitrary numbers. By varying them, we obtain different solutions to (18)

5 Exact solution for Burgers-Huxley equation

We consider the Burgers-Huxley equation

\[
u_t = u_{xx} + uu_x + u(k - u)(u - 1),
\]

(39)
where $k$ is a nonzero real constant. Assume that Eq. (39) has an exact solution in the form

$$u(x, t) = u(\xi), \quad \xi = x - ct,$$

(40)

where $c$ is real constant to be determined. By substituting (40) into Eq. (39), we obtain

$$cu' + uu' + u'' + u(k - u)(u - 1) = 0,$$

(41)

where prime denotes the derivative with respect to the same variable $\xi$.

Next, we introduce new independent variables $x = u$, $y = u_x$ which changes (41) to a system of ODEs

$$\begin{align*}
  x' &= y, \\
  y' &= -cy - xy - x(k - x)(x - 1).
\end{align*}$$

(42)

Now, we apply the Division theorem to seek the first integral to (42). Suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to (42), and $p(x, y) = \sum_{i=0}^{m} a_i(x) y^i$, is an irreducible polynomial in $\mathbb{C}[x, y]$ such that

$$p(x(\xi), y(\xi)) = \sum_{i=0}^{m} a_i(x(\xi)) y(\xi)^i = 0,$$

(43)

where $a_i(x)$ ($i = 0, \cdots, m$) are polynomials of $x$ and $a_m(x) \neq 0$. We start our study by assuming $m = 1$ in (43). By the Division Theorem, there exists a polynomial $H(x, y) = h(x) + g(x) y$ in $\mathbb{C}[x, y]$ such that

$$\begin{align*}
  \frac{dp}{d\xi}(42) &= \left( \frac{\partial p}{\partial x} \frac{\partial x}{\partial\xi} + \frac{\partial p}{\partial y} \frac{\partial y}{\partial\xi} \right)(42) \\
  &= \sum_{i=0}^{1} a_i'(x) y^{i+1} + \sum_{i=0}^{1} i a_i(x) y^{i-1} [-cy - xy - x(k - x)(x - 1)] \\
  &= (h(x) + g(x) y) \left( \sum_{i=0}^{1} a_i(x) y^i \right),
\end{align*}$$

(44)

where prime denote differentiating with respect to the variable $x$. On equating the coefficients of $y^i$ ($i = 2, 1, 0$) on both sides of (44), we have

$$a_1'(x) = g(x) a_1(x),$$

(45)

$$a_0'(x) = ca_1(x) + xa_1(x) + h(x) a_1(x) + g(x) a_0(x),$$

(46)

$$-a_1(x) x(k - x)(x - 1) = h(x) a_0(x).$$

(47)
Since, $a_1(x)$ is a polynomial of $x$, from (45) we conclude that $a_1(x)$ is a constant and $g(x) = 0$. For simplicity, we take $a_1(x) = 1$, and balancing the degrees of $h(x)$ and $a_0(x)$ we conclude that $\deg h(x) = 1$, only. Now, suppose that $h(x) = Ax + B$, then from (46) we find

\[ a_0(x) = \frac{(A + 1)}{2}x^2 + (B + c)x + D, \]

where $D$ is an arbitrary integration constant. Substituting $a_0(x)$, $a_1(x)$ and $h(x)$ in (47) and setting the coefficient of powers $x$ to be zero, then yields

\[
\begin{cases}
\frac{A(A + 1)}{2} = 1, \\
A(B + c) + \frac{B(A + 1)}{2} = -k - 1, \\
AD + B(B + c) = k, \\
BD = 0.
\end{cases}
\]

(48)

Solving (48), we obtain

\[
A = 1, \quad B = 0, \quad D = k, \quad c = -k - 1,
\]

(49)

\[
A = 1, \quad B = -k, \quad D = 0, \quad c = k - 1,
\]

(50)

\[
A = 1, \quad B = -1, \quad D = 0, \quad c = 1 - k,
\]

(51)

\[
A = -2, \quad B = 0, \quad D = -\frac{k}{2}, \quad c = \frac{k + 1}{2},
\]

(52)

\[
A = -2, \quad B = 2, \quad D = 0, \quad c = \frac{k - 4}{2},
\]

(53)

\[
A = -2, \quad B = 2k, \quad D = 0, \quad c = \frac{-4k + 1}{2}.
\]

(54)

Setting (49)-(54) in (43), we obtain

\[
y + x^2 - (k + 1)x + k = 0,
\]

\[
y + x^2 - x = 0,
\]

\[
y + x^2 - kx = 0,
\]

\[
y - \frac{1}{2}x^2 + \frac{k + 1}{2}x - \frac{k}{2} = 0,
\]

(55)

\[
y - \frac{1}{2}x^2 + \frac{k}{2}x = 0,
\]

\[
y - \frac{1}{2}x^2 + \frac{1}{2}x = 0,
\]
respectively. Combining equations (55) with (42), we find the exact solutions to equation (41) as follows

\[ u_1(\xi) = \frac{ke^{(k-1)(\xi+c_1)} - 1}{e^{(k-1)(\xi+c_1)} - 1}, \]

\[ u_2(\xi) = \frac{1}{1 + c_1 e^{-\xi}}, \]

\[ u_3(\xi) = \frac{k}{1 + c_1 ke^{(-k\xi)}}, \]

\[ u_4(\xi) = \frac{e^{\frac{1}{2}(k-1)(\xi+c_1)} - k}{e^{\frac{1}{2}(k-1)(\xi+c_1)} - 1}, \]

\[ u_5(\xi) = \frac{k}{1 + c_1 ke^{\left(\frac{k\xi}{2}\right)}}, \]

\[ u_6(\xi) = \frac{1}{1 + c_1 e^{\frac{\xi}{2}}}, \]

where \( c_1 \) is an arbitrary constant. Then the exact solutions to (39) can be written as

\[ u_1(x, t) = \frac{ke^{(k-1)(x+(k+1)t+c_1)} - 1}{e^{(k-1)(x+(k+1)t+c_1)} - 1}, \]

\[ u_2(x, t) = \frac{1}{1 + c_1 e^{-x+(k-1)t}}, \]

\[ u_3(x, t) = \frac{k}{1 + c_1 ke^{(-kx+k(1-k)t)}}, \]

\[ u_4(x, t) = \frac{e^{\frac{1}{2}(k-1)(x-(\frac{k+1}{2})t+c_1)} - k}{e^{\frac{1}{2}(k-1)(x-(\frac{k+1}{2})t+c_1)} - 1}, \]

\[ u_5(x, t) = \frac{k}{1 + c_1 ke^{\left(\frac{kx-(k-4)c_1}{2}\right)t}}, \]

\[ u_6(x, t) = \frac{1}{1 + c_1 e^{\frac{x}{2}}}. \]
In these solutions, \( c \) is arbitrary numbers. By varying it, we obtain different solutions to (39)

6 Exact solution for Newell-Whitehead equation

Consider the Newell-Whitehead equation

\[
    u_t - u_{xx} = \beta u(1 - u)(u+1),
\]

where \( \beta \) is a nonzero real constant. Assume that Eq. (56) has an exact solution in the form

\[
    u(x, t) = u(\xi), \quad \xi = x - ct,
\]

where \( c \) is real constant to be determined. By substituting (57) into Eq. (56), we obtain

\[
    cu' + u'' + \beta u(1 - u)(u + 1) = 0,
\]

where prime denotes the derivative with respect to the same variable \( \xi \).

Next, we introduce new independent variables \( x = u, y = u_\xi \) which changes (58) to a system of ODEs

\[
    \begin{cases}
        x' = y, \\
        y' = -cy - \beta x(1 - x^2).
    \end{cases}
\]

Now, we apply the Division theorem to seek the first integral to (59). Suppose that \( x = x(\xi) \) and \( y = y(\xi) \) are the nontrivial solutions to (59), and

\[
p(x, y) = \sum_{i=0}^{m} a_i(x)y^i,
\]

is an irreducible polynomial in \( \mathbb{C}[x, y] \) such that

\[
    p(x(\xi), y(\xi)) = \sum_{i=0}^{m} a_i(x(\xi))y(\xi)^i = 0,
\]

where \( a_i(x) \) \( (i = 0, \cdots, m) \) are polynomials of \( x \) and \( a_m(x) \neq 0 \). We start our study by assuming \( m = 1 \) in (60). By the Division Theorem, there exists a polynomial \( H(x, y) = h(x) + g(x)y \) in \( \mathbb{C}[x, y] \) such that

\[
    \left. \frac{dp}{d\xi} \right|_{(59)} = \left( \frac{\partial p}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y} \frac{\partial y}{\partial \xi} \right) |_{(59)}
\]

\[
= \sum_{i=0}^{1} a'_i(x)y^{i+1} + \sum_{i=0}^{1} ia_i(x)y^{i-1}[-cy - \beta x(1 - x^2)]
\]

\[
= (h(x) + g(x)y) \left( \sum_{i=0}^{1} a_i(x)y^i \right),
\]
where prime denote differentiating with respect to the variable $x$. On equating the coefficients of $y^i$ ($i = 2, 1, 0$) on both sides of (61), we have

$$a'_1(x) = g(x)a_1(x), \quad (62)$$
$$a'_0(x) = c a_1(x) + h(x)a_1(x) + g(x)a_0(x), \quad (63)$$
$$-\beta a_1(x)x(1-x^2) = h(x)a_0(x). \quad (64)$$

Since, $a_1(x)$ is a polynomial of $x$, from (62) we conclude that $a_1(x)$ is a constant and $g(x) = 0$. For simplicity, we take $a_1(x) = 1$, and balancing the degrees of $h(x)$ and $a_0(x)$ we conclude that $\text{deg } h(x) = 1$, only. Now, suppose that $h(x) = Ax + B$, then from (63) we find

$$a_0(x) = \frac{(A)}{2} x^2 + (B + c) x + D,$$

where $D$ is an arbitrary integration constant. Substituting $a_0(x), a_1(x)$ and $h(x)$ in (64) and setting the coefficient of powers $x$ to be zero, then yields

$$\begin{aligned}
\frac{A^2}{2} &= \beta, \\
\frac{3AB}{2} + Ac &= 0, \\
AD + B(B + c) &= -\beta, \\
BD &= 0.
\end{aligned} \quad (65)$$

Solving (65), we obtain

$$A = \sqrt{2\beta}, \quad B = -\sqrt{2\beta}, \quad D = 0, \quad c = 3\sqrt{\frac{\beta}{2}}, \quad (66)$$
$$A = \sqrt{2\beta}, \quad B = \sqrt{2\beta}, \quad D = 0, \quad c = -3\sqrt{\frac{\beta}{2}}, \quad (67)$$
$$A = -\sqrt{2\beta}, \quad B = -\sqrt{2\beta}, \quad D = 0, \quad c = 3\sqrt{\frac{\beta}{2}}, \quad (68)$$
$$A = -\sqrt{2\beta}, \quad B = \sqrt{2\beta}, \quad D = 0, \quad c = -3\sqrt{\frac{\beta}{2}}. \quad (69)$$
Setting \((66)-(69)\) in \((60)\), we obtain

\[
y + \sqrt{\frac{\beta}{2}} x^2 + \sqrt{\frac{\beta}{2}} x = 0,
\]
\[
y + \sqrt{\frac{\beta}{2}} x^2 - \sqrt{\frac{\beta}{2}} x = 0,
\]
\[
y - \sqrt{\frac{\beta}{2}} x^2 + \sqrt{\frac{\beta}{2}} x = 0,
\]
\[
y - \sqrt{\frac{\beta}{2}} x^2 - \sqrt{\frac{\beta}{2}} x = 0,
\]

respectively. Combining equations \((70)\) with \((59)\), we find the exact solutions to equation \((58)\) as follows

\[
u_1(\xi) = \frac{1}{c_1 e^{\frac{\sqrt{2\beta}}{2} \xi} - 1},
\]
\[
u_2(\xi) = \frac{1}{c_1 e^{-\frac{\sqrt{2\beta}}{2} \xi} + 1},
\]
\[
u_3(\xi) = \frac{1}{c_1 e^{\frac{\sqrt{2\beta}}{2} \xi} + 1},
\]
\[
u_4(\xi) = \frac{1}{c_1 e^{-\frac{\sqrt{2\beta}}{2} \xi} - 1},
\]

where \(c_1\) is an arbitrary constant. Then the exact solutions to \((56)\) can be written as

\[
u_1(x, t) = \frac{1}{c_1 e^{\frac{\sqrt{2\beta}}{2} (x-3\sqrt{\frac{\beta}{2}} t)} - 1},
\]
\[
u_2(x, t) = \frac{1}{c_1 e^{-\frac{\sqrt{2\beta}}{2} (x+3\sqrt{\frac{\beta}{2}} t)} + 1},
\]
\[
u_3(x, t) = \frac{1}{c_1 e^{\frac{\sqrt{2\beta}}{2} (x-3\sqrt{\frac{\beta}{2}} t)} + 1},
\]
\[
u_4(x, t) = \frac{1}{c_1 e^{-\frac{\sqrt{2\beta}}{2} (x+3\sqrt{\frac{\beta}{2}} t)} - 1}.
\]

In these solutions, \(c\) is arbitrary numbers. By varying it, we obtain different solutions to \((56)\)
7 Conclusions:

In this paper, an implementation of the first integral method is given by applying it to four nonlinear equations to illustrate the validity and advantages of the method. The obtained solutions with free parameters may be important to explain some physical phenomena.

References


